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Large *N* Penner matrix model and a novel asymptotic formula for the generalized Laguerre polynomials

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Abstract

The Gaussian Penner matrix model is re-examined in the light of the results which have been found in double-well matrix models. The orthogonal polynomials for the Gaussian Penner model are shown to be the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ with α and x depending on N, the size of the matrix. An asymptotic formula for the orthogonal polynomials is derived following closely the orthogonal polynomial method of Deo (1997 *Nucl. Phys.* B **504** 609). The universality found in the double-well matrix model is extended to include non-polynomial potentials. An asymptotic formula is also found for the Laguerre polynomial using the saddle-point method by rescaling α and x with N. Combining these results a novel asymptotic formula is found for the generalized Laguerre polynomials (different from that given in Szego's book) in a different asymptotic regime. This may have applications in mathematical and physical problems in the future. The density–density correlators are derived and are the same as those found for the double-well matrix models. These correlators in the smoothed large N limit are sensitive to odd and even N where N is the size of the matrix. These results for the two-point density–density correlation function may be useful in finding eigenvalue effects in experiments in mesoscopic systems or small metallic grains. There may be applications to string theory as well as the tunnelling of an eigenvalue from one valley to the other being an important quantity there.

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1. Introduction

In this work we revisit the generalized Penner model [1] and its associated Laguerre polynomials. The Gaussian Penner model is a double-well matrix model with an infinite thin barrier at the origin, hence its eigenvalue spectrum has a gap at the origin. Here a novel asymptotic formula for the orthogonal polynomials (generalized Laguerre polynomials) of the Gaussian Penner model is found. This allows the basic kernel of the model, using the Christoffel–Darboux formula, to be derived exactly. Once the kernel is known the density, density–density correlators and higher order correlators, which are the observables of the model, can be determined. See [2] for the results found for a general double-well matrix model. The work of [6] which involved the loop equation method and [2–4] which used the orthogonal polynomial method, a variation of the loop equation method and the method due to Shohat, gave different results for the smoothed long range density–density correlators. This was clarified in [5] where it has been argued that the difference is due to not taking into account the discreteness of the spectrum. Taking into account the discreteness leads to an extra term which is beyond the mean field result [6]. Combining these gives rise to the result of [2–4]. In the mathematics literature there have also been studies of the double-well matrix model [7]. The interest from mathematicians has been primarily because these models fall into the class of oscillatory Riemann–Hilbert problems as the recurrence coefficients of these models are highly oscillatory. In the case of the double-well matrix models the problem can be exactly analytically solved.

The Penner matrix model was also studied in the context of the moduli space of a punctured surface [8]. There an equality between the Penner matrix model and the Euler characteristic of moduli space of punctured surfaces was computed, before taking the continuum limit. Later in [9] it was shown that even after taking the continuum limit the Euler characteristic of moduli space of unpunctured surfaces was obtained as the free energy of the Penner model. This was done in an effort to understand two-dimensional gravity coupled to matter at the critical point $c = 1$. In [1] the Gaussian Penner model and its multicritical behaviour was studied. Here the study has been extended to derive the correlators of the Gaussian Penner model for large N. The density–density correlators of this model correspond to the probability of finding an eigenvalue given the probability of finding the first eigenvalue. The results obtained here for the long range density–density correlator [2] will give information about the tunnelling of eigenvalues from one valley to the other. In a recent advance [10] it was shown that the properties of string theory may be found by calculating single electron tunnelling in multi-cut matrix models. The work in this paper may have some bearing on the work in [10].

Applications of this study may also be to disordered mesoscopic systems where generalized Laguerre polynomials arise, for example, in disordered models of metal–insulator transitions [11, 12] and superconducting–normal interface with Andreev reflection (the opening of a gap) [13]. In models of structural glasses [14] which map onto a variant of the Gaussian Penner matrix model these results may be of particular interest. In another direction there has been recent work on the generalized Laguerre ensemble in the context of the chiral random matrix models of QCD (see [15, 16] and references therein) and in describing a novel group structure associated with scattering in disordered mesoscopic wires [12]. The results obtained here may be relevant for such studies in the future. The Penner model results found here are also interesting in their own right.

This study focuses on the Gaussian Penner model [1] with potential $V(M) = \frac{1}{2}\mu M^2 - \frac{t}{2}\ln M^2$ where M is an $N \times N$ random matrix; after some work on gapped matrix models has been reported and clarified [2–5]. In recent work two basic ideas have been implemented. First the idea of symmetry breaking (see [14, 17]) and then an asymptotic formula of the orthogonal polynomials [2, 7]. Here for the Gaussian Penner model the corresponding polynomials are the associate Laguerre polynomials, $L_N^{\alpha}(x)$, where the α, x and N asymptotes are to be taken, in previous work only the x, N asymptotes were found. It turns out that it is in this asymptotic region that the singular models make contact with the double-well random matrix models and

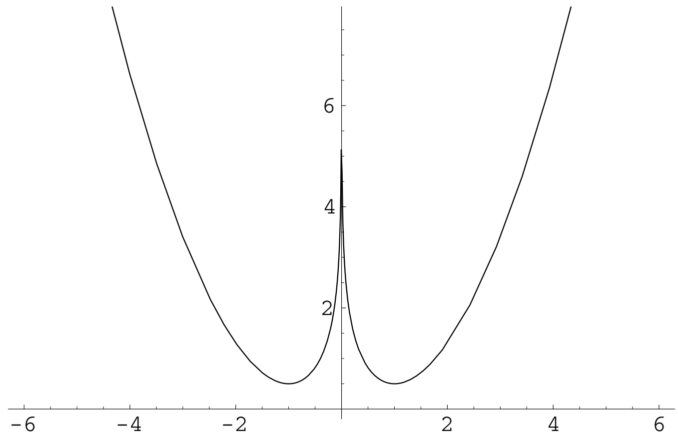

Figure 1. The potential for the Gaussian Penner random matrix model with $V(\lambda) = \frac{1}{2}\mu\lambda^2 - \frac{t}{2}\ln\lambda^2$, $\mu = 1$ and $t = 1$.

the universality of the gapped matrix model is extended to include non-polynomial potentials. The density–density correlators are the same as those found for the double-well matrix models. Ideas of symmetry breaking are also true for the Gaussian Penner model [14, 17] but because of the infinite thin barrier at the origin, and symmetry breaking involves one electron quantum tunnelling, it is harder to give an intuitive picture. These ideas will be elaborated on in future papers.

The paper is divided as follows. It starts by describing completely the model and establishing the notation and conventions. In section 3, the asymptotic formula is derived using saddle point and orthogonal polynomial techniques. This corresponds to the asymptotic formula discussed in [2, 3]. The orthogonal polynomial for the Gaussian Penner model is proportional to the generalized Laguerre polynomials, hence a new asymptotic formula is found for these polynomials. In section 4, the old asymptotic formula of Szego for the associate Laguerre polynomials is taken and shown not to correspond to the asymptotic formula found above. Section 5 ends with conclusions and open questions.

2. The model, notation and conventions

We consider models of the type (see [1] for details of notation and definitions)

$$Z = \exp(-F) = \int \mathrm{d}M\, \mathrm{e}^{-N\,\mathrm{Tr}\,V(M)} \tag{2.1}$$

where $V(M) = V_0(M) - \frac{t}{2}\ln M^2$ and $V_0(M) = \frac{1}{2}\mu M^2$ (see figure 1).

In general the orthogonal polynomials are $P_n(\lambda) = \lambda^n + \text{l.o.}$ where λ are the eigenvalues of M and the orthogonality conditions are $\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-NV(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}$. The partition function can be expressed in terms of the h_n as $Z = N!h_0 h_1 h_2 \cdots h_{N-1}$.

For the large N limit, the density of eigenvalues, $\rho(z) \equiv \left(\frac{1}{N}\right)\sum_{i=1}^{N}\delta(z-\lambda_i)$, can be found by solving either a saddle-point equation or the Schwinger–Dyson equation. In terms of the generating function $F(z) = \frac{1}{N}\left[\mathrm{Tr}\frac{1}{z-M}\right] \to \int \mathrm{d}z' \frac{\rho(z')}{z-z'}$ the Schwinger–Dyson equation reads $F(z)^2 - V'(z)F(z) = M(z)$, where $M(z)$ is a meromorphic function. The density of

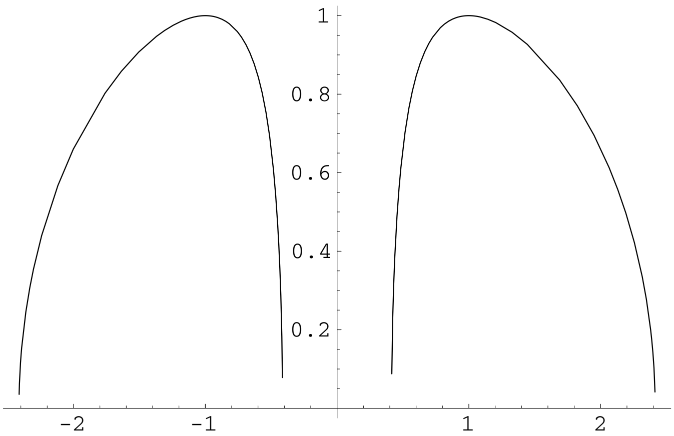

Figure 2. The density of eigenvalues for the Gaussian Penner random matrix model for $\mu = 1$ and $t = 1$.

eigenvalues is given by $\rho(z) = -\frac{1}{\pi} \operatorname{Im} F(z)$. The generating function $F(z)$ in the large N limit for $t > 0$ is

$$F(z) = \frac{\mu z}{2} - \frac{t}{2z} - \frac{\mu}{2z}\sqrt{(z^2 - a^2)(z^2 - b^2)} \tag{2.2}$$

where $a^2 = \frac{(2+t)}{\mu} + \frac{2}{\mu}\sqrt{(1+t)}$ and $b^2 = \frac{(2+t)}{\mu} - \frac{2}{\mu}\sqrt{(1+t)}$. See figure 2 for the corresponding density of eigenvalues.

3. Exact solution for the orthogonal polynomial of the symmetric Gaussian Penner matrix model

The orthogonal polynomials satisfy the recurrence relation

$$(z - S_n)P_n(z) = P_{n+1}(z) + R_n P_{n-1}(z) \tag{3.1}$$

where S_n, R_n are the recurrence coefficients. For symmetric models $S_n = 0$. For even potentials instead of the recurrence relation

$$zP_n(z) = P_{n+1}(z) + R_n P_{n-1}(z) \tag{3.2}$$

one can use

$$z^2 P_{2n}(z) = P_{2n+2}(z) + (R_{2n+1} + R_{2n})P_{2n}(z) + R_{2n-1}R_{2n}P_{2n-2}(z) \tag{3.3}$$

and

$$z^2 P_{2n+1}(z) = P_{2n+3}(z) + (R_{2n+1} + R_{2n+2})P_{2n+1}(z) + R_{2n+1}R_{2n}P_{2n-1}(z) \tag{3.4}$$

where we have multiplied z to equation (3.2) and expanded. Then equation (3.3) contains only even polynomials $P_{2n}(-z) = P_{2n}(z)$ and equation (3.4) contains only odd polynomials $P_{2n+1}(-z) = -P_{2n+1}(z)$. This simplifies the solution as we will see.

(1) Let us first work with the even set. Let $y = z^2$ and define functions $\mathcal{P}_n(y) = P_{2n}(z)$. In terms of these 'new' polynomials

$$y\mathcal{P}_n(y) = \mathcal{P}_{n+1}(y) + \mathcal{S}_n\mathcal{P}_n(y) + \mathcal{R}_n\mathcal{P}_{n-1}(y) \tag{3.5}$$

where $\mathcal{S}_n = R_{2n} + R_{2n+1} = A_n + B_n$ and $\mathcal{R}_n = R_{2n-1}R_{2n} = A_nB_{n-1}$. These polynomials obey

$$\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'[\mathcal{V}_0(y)-t'\log y]}\mathcal{P}_n(y)\mathcal{P}_m(y) = h_n\delta_{n,m} \tag{3.6}$$

where $\mathcal{V}_0(y) = 2V_0(z) = \mu y + \cdots$, $t' = t - \frac{1}{2N'}$ and $N' = \frac{N}{2}$.

This is the same as the brick-wall problem, i.e. the linear Penner model if $t \leftrightarrow t'$, $N \leftrightarrow N'$ and $t'N' = \left(t - \frac{1}{N}\right)\frac{N}{2} = \frac{(Nt-1)}{2}$.

(2) A similar analysis can be carried out for the odd set. Define

$$\bar{\mathcal{P}}_n(y) = z^{-1}P_{2n+1}(z). \tag{3.7}$$

Then

$$y\bar{\mathcal{P}}_n(y) = \bar{\mathcal{P}}_{n+1}(y) + \bar{\mathcal{S}}_n\bar{\mathcal{P}}_n(y) + \bar{\mathcal{R}}_n\bar{\mathcal{P}}_{n-1}(y) \tag{3.8}$$

where

$$\bar{\mathcal{S}}_n = R_{2n+1} + R_{2n+2} \tag{3.9}$$

and

$$\bar{\mathcal{R}}_n = R_{2n+1}R_{2n}. \tag{3.10}$$

Because of the extra factor of z associated with the odd series the 'barred' polynomials satisfy the orthogonality condition

$$\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'[\mathcal{V}_0(y)-\bar{t}'\log y]}\bar{\mathcal{P}}_n(y)\bar{\mathcal{P}}_m(y) = \bar{h}_n\delta_{n,m} \tag{3.11}$$

where $\bar{t}' = t + \frac{1}{2N'}$. This barred system can be solved as a brick-wall problem. The original recurrence coefficients are obtained by

$$R_{2n+1} = \frac{1}{2}\left\{\mathcal{S}_n + \sqrt{\mathcal{S}_n^2 - 4\bar{\mathcal{R}}_n}\right\} \tag{3.12}$$

$$R_{2n} = \frac{1}{2}\left\{\mathcal{S}_n - \sqrt{\mathcal{S}_n^2 - 4\bar{\mathcal{R}}_n}\right\}. \tag{3.13}$$

Using $\mathcal{W}_{n+1} + \mathcal{W}_n + S_n\mathcal{Y}_n = \frac{2n+1+Nt}{N}$ and $S_n\mathcal{W}_{n+1} - \mathcal{W}_n - \frac{1}{N} = -R_{n+1}\mathcal{Y}_{n+1} + R_n\mathcal{Y}_{n-1}$ (see [1] for details and definitions of $\mathcal{W}$ and $\mathcal{Y}$) we get

$$\mathcal{S}_n = \frac{2n+1+t'N'}{\mu N'} = \frac{4n+1+tN}{\mu N} \tag{3.14}$$

$$\bar{\mathcal{S}}_n = \frac{2n+1+\bar{t}'N'}{\mu N'} = \frac{4n+3+tN}{\mu N} \tag{3.15}$$

$$\mathcal{R}_n = \frac{n(n+t'N')}{\mu^2N'^2} = \frac{2n(2n-1+tN)}{\mu^2N^2} \tag{3.16}$$

$$\bar{\mathcal{R}}_n = \frac{n(n+\bar{t}'N')}{\mu^2N'^2} = \frac{2n(2n+1+tN)}{\mu^2N^2}. \tag{3.17}$$

(1) For the even set the orthogonality relation

$$\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'[\mathcal{V}_0(y)-t'\log y]}\mathcal{P}_n(y)\mathcal{P}_m(y) = h_n\delta_{n,m} \tag{3.18}$$

simplifies to

$$\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'\mathcal{V}_0(y)} y^{N't'} \mathcal{P}_n(y)\mathcal{P}_m(y) = h_n\delta_{n,m}. \tag{3.19}$$

Now as $\mathcal{R}_n = \frac{n(n+t'N')}{\mu^2 N'^2} = \frac{2n(2n-1+tN)}{\mu^2 N^2}$ and $\mathcal{R}_n = \frac{h_n}{h_{n-1}}, h_n$ is $h_n = \mathcal{R}_n h_{n-1} = \mathcal{R}_n\mathcal{R}_{n-1}\mathcal{R}_{n-2}\ldots\mathcal{R}_1 h_0 = \frac{n!}{(\mu N')^{2n}}\frac{\Gamma(n+t'N'+1)}{\Gamma(t'N'+1)}h_0$. The integral for h_0 can be solved simply by noting that $\mathcal{P}_n(y) = P_{2n}(z)$ for $n = 0, \mathcal{P}_n(y) = P_0(z) = 1$, then $h_0 = \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'\mathcal{V}_0(y)} y^{N't'}\mathcal{P}_0(y)\mathcal{P}_0(y) = \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'\mathcal{V}_0(y)} y^{N't'}$. For $\mathcal{V}_0(y) = \mu y$ and substituting $u = N'\mu y$ we get $h_0 = \int_0^\infty \frac{\mathrm{d}u}{N'\mu}\mathrm{e}^{-u}\left(\frac{u}{N'\mu}\right)^{N't'} = \frac{1}{(N'\mu)^{N't'+1}}\Gamma(N't'+1)$. Thus $h_n = \frac{n!\Gamma(n+1+t'N')}{(\mu N')^{2n+N't'+1}}$.

Rewriting $\mathcal{P}_n(y) = Q_n(u)$ the orthogonality condition is $\int_0^\infty \mathrm{d}u u^{N't'}\,\mathrm{e}^{-u} Q_n(u)Q_m(u) = (N'\mu)^{N't'+1}h_n\delta_{n,m} = \frac{(n!)^2\Gamma(n+1+t'N')}{(\mu N')^{2n}\Gamma(n+1)}\delta_{n,m}$. Redefine $Q'_n(u) = \frac{(\mu N')^n}{n!}Q_n(u)$ the above integral is

$$\int_0^\infty \mathrm{d}u u^{N't'}\,\mathrm{e}^{-u} Q'_n(u)Q'_n(u) = \frac{\Gamma(n+1+t'N')}{\Gamma(n+1)}\delta_{n,m} \tag{3.20}$$

which on comparing with the corresponding integral for the associate Laguerre polynomials $L_n^{(\alpha)}(x)$ gives $Q'_n(u) = (-1)^n L_n^{N't'}(u)$, then ($\alpha = N't'$ associate Laguerre polynomials) $\mathcal{P}_n(y) = Q_n(u) = \frac{n!}{(\mu N')^n}(-1)^n L_n^{N't'}(u)$ and $\hat{\mathcal{P}}_n(y) = \frac{\mathcal{P}_n(y)}{\sqrt{h_n}} = \sqrt{\frac{(\mu N')^{N't'+1}}{(n!)\Gamma(n+1+t'N')}}(n!)(-1)^n L_n^{N't'}(N'\mu y)$.

Thus the normalized even set of orthogonal polynomials is

$$\begin{aligned}\psi_{2n}(y) &= \mathrm{e}^{-\frac{N'}{2}[\mu y - t'\log y]}\hat{\mathcal{P}}_n(y)\\ &= \left[\frac{n!(N'\mu)^{N't'+1}}{\Gamma(n+1+N't')}\right]^{\frac{1}{2}} y^{\frac{N't'}{2}}\,\mathrm{e}^{\frac{-N'\mu y}{2}} L_n^{N't'}(N'\mu y).\end{aligned} \tag{3.21}$$

(2) For the odd set

$$\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'[\mu y - \bar{t}'\log y]}\bar{\mathcal{P}}_n(y)\bar{\mathcal{P}}_m(y) = \bar{h}_n\delta_{nm} \tag{3.22}$$

where $\bar{t}' = t + \frac{1}{2N'}$ and $N'\bar{t}' = \frac{Nt+1}{2}$. The orthogonality condition is

$$\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'\mu y} y^{N'\bar{t}'}\bar{\mathcal{P}}_n(y)\bar{\mathcal{P}}_m(y) = \bar{h}_n\delta_{nm}. \tag{3.23}$$

Note that $\bar{\mathcal{R}}_n = \frac{\bar{h}_n}{\bar{h}_{n-1}}$, for $\bar{\mathcal{R}}_n = \frac{n(n+\bar{t}')}{\mu^2 N'^2}$ we get $\bar{h}_n = \bar{\mathcal{R}}_n\bar{h}_{n-1} = \bar{\mathcal{R}}_n\bar{\mathcal{R}}_{n-1}\bar{\mathcal{R}}_{n-2}\ldots\bar{\mathcal{R}}_1\bar{h}_0 = \frac{n!(n+\bar{t}'N')!}{(\mu N')^{2n}(\bar{t}'N')!}\bar{h}_0$. Further $\bar{\mathcal{P}}_n(y) = z^{-1}P_{2n+1}(z)$ and $\bar{\mathcal{P}}_0(y) = z^{-1}P_1(z)$ with $P_1(z) = z, \bar{\mathcal{P}}_0(y) = z^{-1}z = 1$. Therefore, the integral for $\bar{h}_0$ is $\bar{h}_0 = \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'\mu y} y^{N'\bar{t}'}\bar{\mathcal{P}}_0(y)\bar{\mathcal{P}}_0(y) = \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-N'\mu y} y^{N'\bar{t}'}$. Defining $u = N'\mu y$, we get $\bar{h}_0 = \int_0^\infty \frac{\mathrm{d}u}{N'\mu}\mathrm{e}^{-u}\left(\frac{u}{N'\mu}\right)^{N'\bar{t}'} = \frac{1}{(N'\mu)^{N'\bar{t}'+1}}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u}u^{N'\bar{t}'}$ so that $\bar{h}_n = \frac{\Gamma(n+1)\Gamma(n+1+\bar{t}'N')}{(N'\mu)^{2n+N'\bar{t}'+1}}$.

Now let $\bar{Q}_n(u) = \bar{\mathcal{P}}_n(y)$. Therefore, $\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u}u^{N'\bar{t}'}\bar{Q}_n(u)\bar{Q}_m(u) = (N'\mu)^{N'\bar{t}'+1}\bar{h}_n\delta_{nm} = \left(\frac{n!}{(\mu N')^n}\right)^2\frac{\Gamma(n+1+\bar{t}'N')}{n!}\delta_{nm}$.

Define $\bar{Q}'_n(u) = \frac{(\mu N')^n}{n!}\bar{Q}_n(u)$ then

$$\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u}u^{N'\bar{t}'}\bar{Q}'_n(u)\bar{Q}'_m(u) = \frac{\Gamma(n+1+\bar{t}'N')}{n!}\delta_{nm}. \tag{3.24}$$

Comparing with the associate Laguerre polynomials $\bar{Q}'_n(u) = \frac{(\mu N')^n}{n!}Q_n(u) = (-1)^n L_n^{N'\bar{t}'}(u)$ where $\alpha = N'\bar{t}'$. Now $\bar{\mathcal{P}}_n(y) = \bar{Q}_n(u) = \frac{n!(-1)^n}{(\mu N')^n}L_n^{N'\bar{t}'}(N'\mu y)$ hence $\hat{\bar{\mathcal{P}}}_n(y) = \frac{\bar{\mathcal{P}}_n(y)}{\sqrt{\bar{h}_n}} = \sqrt{\frac{(\mu N')^{N'\bar{t}'+1}n!}{\Gamma(n+1+\bar{t}'N')}}(-1)^n L_n^{N'\bar{t}'}(N'\mu y)$.

Thus the normalized odd orthogonal polynomials are

$$\psi_{2n+1}(y) = \mathrm{e}^{-\frac{N'}{2}[\mu y - \bar{t}' \log y]} \hat{\bar{\mathcal{P}}}_n(y)$$
$$= \left[\frac{n!(N'\mu)^{N'\bar{t}'+1}}{\Gamma(n+1+N'\bar{t}')} \right]^{\frac{1}{2}} y^{\frac{N'\bar{t}'}{2}} \mathrm{e}^{\frac{-N'\mu y}{2}} L_n^{N'\bar{t}'}(N'\mu y). \tag{3.25}$$

The important lesson to learn from this exercise is that $\alpha = \frac{(Nt-(-1)^{n'})}{2}$ for n' an integer which is even or odd and $x = N'\mu y$ for the orthogonal polynomials of the Gaussian Penner model (which are proportional to the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$).

4. A new asymptotic formula for the generalized Laguerre polynomial

Let us start with an expression which has a well-defined $\alpha = 0$ limit. Using

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}u} f(u+x)|_{u=0} \tag{4.1}$$

we can write using Cauchy's theorem

$$L_n^{(\alpha)}(x) = \frac{\mathrm{e}^x}{n!} x^{-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n [(x+u)^{n+\alpha}\,\mathrm{e}^{-(x+u)}]_{u=0}$$
$$= \frac{x^{-\alpha}}{n!} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^n [(x+u)^{(n+\alpha)}\,\mathrm{e}^{-(u)}]|_{u=0}$$
$$= x^{-\alpha} \int_c \frac{\mathrm{d}z}{2\mathrm{i}\pi} \frac{1}{z^{n+1}} (z+x)^{n+\alpha}\,\mathrm{e}^{-z} \tag{4.2}$$

in which the contour C is a small circle around the origin ($|z| < x$). Change $z \to nz$ $\left(|z| < \frac{x}{n}\right)$ then

$$L_n^{(\alpha)}(x) = \left(\frac{x}{n}\right)^{-\alpha} \int_c \frac{\mathrm{d}z}{2\mathrm{i}\pi} \frac{1}{z^{n+1}} \left(z + \frac{x}{n}\right)^{n+\alpha} \mathrm{e}^{-nz}$$
$$= \left(\frac{x}{n}\right)^{-\alpha} \int_c \frac{\mathrm{d}z}{2\mathrm{i}\pi} \frac{1}{z}\,\mathrm{e}^{-nf(z)} \tag{4.3}$$

$$f(z) = z + \log z - \left(1 + \frac{\alpha}{n}\right) \log\left(z + \frac{x}{n}\right). \tag{4.4}$$

We explore $\frac{\alpha}{n}$ and $\frac{x}{n}$ finite. In this representation the limit $\alpha = 0$ is well defined. C is a circle around the origin. The saddle point is given by an expression

$$z^2 + z\left(\frac{x-\alpha}{n}\right) + \frac{x}{n} = 0. \tag{4.5}$$

With the parametrization

$$\frac{\alpha - x}{n} = 2\sqrt{\frac{x}{n}} \cos\phi \tag{4.6}$$

$$z_0 = \sqrt{\frac{x}{n}}\,\mathrm{e}^{\mathrm{i}\phi} \tag{4.7}$$

and $\bar{z}_0$ are the saddle points. This is valid in the range $\left|\frac{\alpha-x}{n}\right| < 2\sqrt{\frac{x}{n}}$; if the parameters are such that this inequality is reversed, there is only one real saddle point. This is where the new saddle points will have to be taken into account.

If there is a saddle point in the integral

$$I = \int_c \frac{\mathrm{d}z}{2\mathrm{i}\pi} \frac{1}{z} \mathrm{e}^{-nf(z)} \tag{4.8}$$

at z_0 then

$$I \approx \frac{\mathrm{e}^{-nf(z_0)}}{z_0} \sqrt{\frac{2\pi}{n|f''(z_0)|}} \mathrm{e}^{\frac{-\mathrm{i}\theta}{2}} \tag{4.9}$$

with $f''(z_0) = |f''(z_0)|\,\mathrm{e}^{\mathrm{i}\theta}$. Expand $f(z)$ around z_0

$$f(z) = f(z_0) + \tfrac{1}{2}(z - z_0)^2 |f''(z_0)|\,\mathrm{e}^{\mathrm{i}\theta}. \tag{4.10}$$

The path is $z - z_0 = x\,\mathrm{e}^{\frac{-\mathrm{i}\theta}{2}}$ and

$$I \approx \mathrm{e}^{-\mathrm{i}\frac{\theta}{2} - nf(z_0)} \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{n}{2}x^2|f''(z_0)|}. \tag{4.11}$$

Here we need to add the contributions of z_0, $\bar{z}_0$ and the other saddle points. Therefore, if $f(z_0) = A + \mathrm{i}B$

$$I \approx \mathrm{e}^{-nA} \sqrt{\frac{2\pi}{x|f''(z_0)|}} 2\cos\left(nB + \phi + \frac{\theta}{2}\right)(x) \tag{4.12}$$

with

$$f''(z_0) = \sqrt{\frac{n}{x}} \frac{2\mathrm{i}\sin\phi\,\mathrm{e}^{-\mathrm{i}\phi}}{\sqrt{\frac{x}{n}} + \mathrm{e}^{\mathrm{i}\phi}} \tag{4.13}$$

and θ is its phase. The $(-1)^n$ would only show up in B. This is a new asymptotic expansion of the generalized Laguerre ensemble in a novel asymptotic regime. It would be nice to have a physical picture of this asymptotic regime.

We can also follow [2] and derive the asymptotic formula for the orthogonal polynomials of the Gaussian Penner random matrix model. A brief derivation is given after the result is stated. For N large but $N - n \approx O(1)$ and x lying in the two cuts the asymptotic formula for the orthogonal polynomials of the Gaussian Penner matrix model can be approximated by

$$\psi_n(x) = \frac{1}{\sqrt{f}} \left[\cos(N\zeta - (N-n)\phi + \chi + (-1)^n\eta)(x) + O\left(\frac{1}{N}\right)\right] \tag{4.14}$$

where f, ζ, ϕ, χ and η are functions of x and are given by

$$\begin{aligned}
f(x) &= \frac{\pi}{2x} \frac{(b^2 - a^2)}{2} \sin 2\phi(x) \\
\zeta'(x) &= -\pi\rho(x) \\
\cos 2\phi(x) &= \frac{x^2 - \frac{(a^2+b^2)}{2}}{\frac{(b^2-a^2)}{2}} \\
\cos 2\eta(x) &= b\frac{\cos\phi(x)}{x} \\
\sin 2\eta(x) &= a\frac{\sin\phi(x)}{x} \\
\chi(x) &= \frac{1}{2}\phi(x) - \frac{\pi}{4}
\end{aligned} \tag{4.15}$$

with a^2 and b^2 as given in section 2 for the Gaussian Penner model.

A brief derivation of this result is given here for completeness. Using $\psi_n(\lambda) = \frac{P_n(\lambda)}{\sqrt{h_n}} \exp\left(-\frac{N}{2}\,\mathrm{Tr}\,V(\lambda)\right)$, equation (3.1) can be written as

$$\lambda\psi_n(\lambda) = \sqrt{R_{n+1}}\psi_{n+1}(\lambda) + \sqrt{R_n}\psi_{n-1}(\lambda). \tag{4.16}$$

Multiplying by λ and using equation (4.16) once again we get

$$\lambda^2\psi_n(\lambda) = \sqrt{R_{n+1}}\sqrt{R_{n+2}}\psi_{n+2}(\lambda) + R_{n+1}\psi_n(\lambda) + R_n\psi_n(\lambda) + \sqrt{R_n}\sqrt{R_{n-1}}\psi_{n-2}(\lambda). \tag{4.17}$$

For potential V with two symmetric wells, it is known that $R_n = A_n$ for n even and $R_n = B_n$ for n odd, where in the large N limit A_n and B_n are approximated by two continuous functions $A(x)$, $B(x)$ with $x = \frac{n}{N}$. On using the asymptotic formula (4.14) we get

$$\frac{\lambda^2 - (A+B)}{2\sqrt{AB}} = \cos 2\phi(\lambda). \tag{4.18}$$

It is known that for Z_2 symmetric two-cut models the endpoints a and b are related to A and B by

$$A + B = \frac{a^2+b^2}{2} \qquad 2\sqrt{AB} = \frac{b^2-a^2}{2}. \tag{4.19}$$

Next consider the recurrence relations (n even)

$$\begin{aligned} \lambda\psi_{n+1}(\lambda) &= \sqrt{R_{n+2}}\psi_{n+2}(\lambda) + \sqrt{R_{n+1}}\psi_n(\lambda) \\ \lambda\psi_{n-1}(\lambda) &= \sqrt{R_n}\psi_n(\lambda) + \sqrt{R_{n-1}}\psi_{n-2}(\lambda). \end{aligned} \tag{4.20}$$

On substituting equation (4.14) for ψ_n, $\psi_{n\pm1}$, $\psi_{n\pm2}$ in equation (4.20) it is easy to see that

$$\begin{aligned} \sin 2\eta(\lambda) &= (-\sqrt{A} + \sqrt{B})\frac{\sin\phi(\lambda)}{\lambda} \\ &= a\frac{\sin\phi(\lambda)}{\lambda} \\ \cos 2\eta(\lambda) &= (\sqrt{A} + \sqrt{B})\frac{\cos\phi(\lambda)}{\lambda} \\ &= b\frac{\cos\phi(\lambda)}{\lambda}. \end{aligned} \tag{4.21}$$

These relations determine some of the equations in (4.15). The rest of the equations can be determined by the orthogonality and expressions for the kernel following closely [2].

Recall that the normalized orthogonal polynomial for this model is $\psi_{2n}(y) = \left[\frac{n!(N'\mu)^{N't'+1}}{\Gamma(n+1+N't')}\right]^{\frac{1}{2}} y^{\frac{N't'}{2}}\,\mathrm{e}^{\frac{-N'\mu y}{2}} L_n^{N't'}(N'\mu y)$ and $\psi_{2n+1}(y) = \left[\frac{n!(N'\mu)^{N'\bar{t}'+1}}{\Gamma(n+1+N'\bar{t}')}\right]^{\frac{1}{2}} y^{\frac{N'\bar{t}'}{2}}\,\mathrm{e}^{\frac{-N'\mu y}{2}} L_n^{N'\bar{t}'}(N'\mu y)$. Hence comparing with equation (4.14) we get a new asymptotic formula for the generalized Laguerre polynomials.

All the corresponding correlation functions of this model will be as obtained in [2] and [3]. Following [3] and using the contour of integration for the Gaussian Penner model, see figure 3, the smoothed density–density correlation function can be derived in the thermodynamic limit and is an oscillating function of N:

$$\begin{aligned} 2\pi^2N^2\rho_2^c(\lambda,\mu) = {}& \frac{\epsilon_\lambda\epsilon_\mu}{\beta\sqrt{|\sigma(\lambda)|}\sqrt{|\sigma(\mu)|}}\frac{1}{(\mu-\lambda)^2}(\lambda\mu(\lambda\mu - a^2 - b^2) \\ & + a^2b^2 + (-1)^N ab(\mu-\lambda)^2). \end{aligned} \tag{4.22}$$

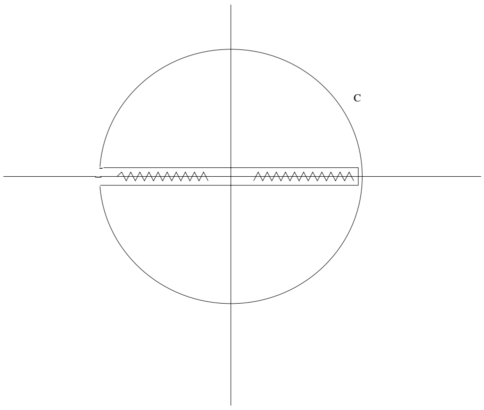

Figure 3. The contour of integration for the two-cut Gaussian Penner model.

Here for the symmetric Gaussian Penner model $\sigma(z) = (z^2 - a^2)(z^2 - b^2)$, $a^2 = \frac{(2+t)}{\mu} + \frac{2}{\mu}\sqrt{(1+t)}$, $b^2 = \frac{(2+t)}{\mu} - \frac{2}{\mu}\sqrt{(1+t)}$, $\epsilon_\lambda = +1$ for $b < \lambda < a$, $\epsilon_\lambda = -1$ for $-a < \lambda < -b$ and $\beta = 1, 2, 4$ depending on whether the matrix M is real orthogonal, Hermitian or self-dual quartonian.

The result for the long range density–density correlation function derived in [3] is for eigenvalues which are not frozen in the two valleys. Using the orthogonal polynomial method of [2] in section V of [3] the result (4.22) is derived; these are the smoothed density–density correlators which correspond to frozen eigenvalues equally distributed in both wells. In [2] the full kernel for the density–density correlator is presented with all the oscillations and dependence on N. The result (4.22) has been obtained by several authors, in particular in [4, 5] where the contradiction between the result in [6] and equation (4.22) was clarified as being due to the discreteness of the eigenvalue spectrum.

As the density–density correlator contains information about the probability of finding an eigenvalue given the probability of finding the first eigenvalue, it contains information about tunnelling between the two valleys if the eigenvalues are in two different wells. Thus the result (4.22) may be useful in the studies of [10] where the significance to string theory of tunnelling from one well to the other in multi-cut matrix models is made.

There may be applications of these expressions for the density–density correlators in the formulae for the mesoscopic fluctuation for situations where there is a gap in the spectrum. The conductance fluctuation for such systems would then depend on N. These may be observed in single electron experiments on mesoscopic samples which have gaps in their eigenvalue spectrum. Work to explicitly obtain all these formulae in the Penner matrix model in this asymptotic regime is in progress. Note that in [18] the application of the smoothed correlators of the single-well matrix model [19] to conductance fluctuations of mesoscopic systems is given.

5. The old asymptotic formula for the generalized Laguerre polynomials

Consider the asymptotic formula given in Szego's book on orthogonal polynomials. The formulae of Plancherel–Rotach type for Laguerre polynomials for α arbitrary and real, ϵ fixed positive number for $x = (4n + 2\alpha + 2)\cos^2\phi$, $\epsilon \leqslant \phi \leqslant \frac{\pi}{2} - \epsilon n^{\frac{-1}{2}}$ are given as

$$e^{\frac{-x}{2}} L_n^{(\alpha)}(x) = (-1)^n (\pi \sin\phi)^{\frac{-1}{2}} x^{-\frac{\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \times \left\{ \cos\left[\left(n + \frac{(\alpha+1)}{2} \right)(\sin 2\phi - 2\phi) + \frac{\pi}{4} \right] + (nx)^{\frac{-1}{2}} O(1) \right\}. \tag{5.1}$$

For the Gaussian Penner model $\alpha = \frac{(Nt-(-1)^{n'})}{2}$ and $x = N'\mu y$ hence the above expression is

$$e^{\frac{-x}{2}} L_n^{(\alpha)}(x) = (-1)^n (\pi \sin\phi)^{\frac{-1}{2}} x^{\frac{-\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \left\{ \cos\left[\left(n + \frac{1}{2} \right)(\sin 2\phi - 2\phi) + \frac{(Nt - (-1)^{n'})}{4}(\sin 2\phi - 2\phi) + \frac{\pi}{4} \right] + (nx)^{\frac{-1}{2}} O(1) \right\}. \tag{5.2}$$

Simplifying we get

$$e^{\frac{-x}{2}} L_n^{(\alpha)}(x) = (-1)^n (\pi \sin\phi)^{\frac{-1}{2}} x^{\frac{-\alpha}{2}-\frac{1}{4}} n^{\frac{\alpha}{2}-\frac{1}{4}} \left\{ \cos\left[\left(n + \frac{1}{2} + \frac{Nt}{4} \right)(\sin 2\phi - 2\phi) + \frac{\pi}{4} - \frac{(-1)^{n'}}{4}(\sin 2\phi - 2\phi) \right] + (nx)^{-\frac{1}{2}} O(1) \right\}. \tag{5.3}$$

Note that the term $\frac{(-1)^{n'}}{4}(\sin 2\phi - 2\phi)$ is not the extra term (with η see [2, 3, 7] or equation (4.15)) that was found for the asymptotic formula in the double-well matrix problem.

6. Conclusion

Let us note that in matrix models when the number of connected components for the support of the eigenvalues changes, one finds a new universality class for the correlators which has been extended here to include the non-polynomial potentials. The orthogonal polynomial for the Gaussian Penner model is derived explicitly and found to be proportional to the generalized Laguerre polynomials $L_n^{\alpha}(x)$ with $\alpha = \frac{(Nt-(-1)^{n'})}{2}$ and $x = N'\mu y$ where n' stands for both even $2n$ and odd $2n+1$. The asymptotic formula found for the orthogonal polynomials of the double-well matrix models is extended to include the Gaussian Penner matrix models. Comparing the above, a novel asymptotic formula for the generalized Laguerre polynomials in a different asymptotic regime is found. As the generalized Laguerre polynomials appear in many applications to mesoscopic systems some of which are mentioned in the introduction, there may be many physical applications in the future. There may be applications to string theory as well as the tunnelling of an eigenvalue from one valley to the other being an important quantity. The density–density correlator presented here may be of relevance in that context. Mathematically, this is also an interesting result as the generalized Penner model are singular models. Moreover a new asymptotic formula for the generalized Laguerre polynomial has been found. Further mathematical and physical relevance of these results will be reported elsewhere.

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